

# Mixed spatially varying $L^2$ -BV regularization of inverse ill-posed problems

Gisela L. Mazziari\*      Ruben D. Spies<sup>✉,†</sup>      Karina G. Temperini<sup>‡</sup>

March 25, 2014

## Abstract

Several generalizations of the traditional Tikhonov-Phillips regularization method have been proposed during the last two decades. Many of these generalizations are based upon inducing stability throughout the use of different penalizers which allow the capturing of diverse properties of the exact solution (e.g. edges, discontinuities, borders, etc.). However, in some problems in which it is known that the regularity of the exact solution is heterogeneous and/or anisotropic, it is reasonable to think that a much better option could be the simultaneous use of two or more penalizers of different nature. Such is the case, for instance, in some image restoration problems in which preservation of edges, borders or discontinuities is an important matter. In this work we present some results on the simultaneous use of penalizers of  $L^2$  and of bounded variation (BV) type. For particular cases, existence and uniqueness results are proved. Open problems are discussed and results to signal restoration problems are presented.

## 1 Introduction and preliminaries

For our general setting we consider the problem of finding  $u$  in an equation of the form

$$Tu = v, \tag{1}$$

where  $T : \mathcal{X} \rightarrow \mathcal{Y}$  is a bounded linear operator between two infinite dimensional Hilbert spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , the range of  $T$  is non-closed and  $v$  is the data, which is supposed to be known, perhaps with a certain degree of error. In the sequel and unless otherwise specified, the space  $\mathcal{X}$  will be  $L^2(\Omega)$  where  $\Omega \subset \mathbb{R}^n$  is a bounded open convex set with Lipschitz boundary. It is well known that under these hypotheses problem (1) is ill-posed in the sense of Hadamard ([8]) and

---

\*Instituto de Matemática Aplicada del Litoral, IMAL, CONICET-UNL, Güemes 3450, S3000GLN, Santa Fe, Argentina, Departamento de Matemática, Facultad de Bioquímica y Ciencias Biológicas, Universidad Nacional del Litoral, Santa Fe, Argentina (gmazziari@santafe-conicet.gov.ar).

†Instituto de Matemática Aplicada del Litoral, IMAL, CONICET-UNL, Güemes 3450, S3000GLN, Santa Fe, Argentina and Departamento de Matemática, Facultad de Ingeniería Química, Universidad Nacional del Litoral, Santa Fe, Argentina (✉: rspies@santafe-conicet.gov.ar).

‡Instituto de Matemática Aplicada del Litoral, IMAL, CONICET-UNL, Güemes 3450, S3000GLN, Santa Fe, Argentina, Departamento de Matemática, Facultad de Humanidades y Ciencias, Universidad Nacional del Litoral, Santa Fe, Argentina (ktemperini@santafe-conicet.gov.ar).

it must be regularized before any attempt to approximate its solutions is made ([7]). The most usual way of regularizing a problem is by means of the use of the *Tikhonov-Phillips regularization method* whose general formulation can be given within the context of an unconstrained optimization problem. In fact, given an appropriate penalizer  $W(u)$  with domain  $\mathcal{D} \subset \mathcal{X}$ , the regularized solution obtained by the Tikhonov-Phillips method and such a penalizer, is the minimizer  $u_\alpha$ , over  $\mathcal{D}$ , of the functional

$$J_{\alpha,W}(u) = \|Tu - v\|^2 + \alpha W(u), \quad (2)$$

where  $\alpha$  is a positive constant called regularization parameter. For general penalizers  $W$ , sufficient conditions guaranteeing existence, uniqueness and weak and strong stability of the minimizers under different types of perturbations where found in [11].

Each choice of an appropriate penalizer  $W$  originates a different regularization method producing a particular regularized solution possessing particular properties. Thus, for instance, the choice of  $W(u) = \|u\|_{L^2(\Omega)}^2$  gives raise to the classical Tikhonov-Phillips method of order zero producing always smooth regularized approximations which approximate, as  $\alpha \rightarrow 0^+$ , the best approximate solution (i.e. the least squares solution of minimum norm) of problem (1) (see [7]) while for  $W(u) = \|\nabla u\|_{L^2(\Omega)}^2$  the order-one Tikhonov-Phillips method is obtained. Similarly, the choice of  $W(u) = \|u\|_{BV(\Omega)}$  (where  $\|\cdot\|_{BV}$  denotes the total variation norm) or  $W(u) = \|\nabla u\|_{L^1(\Omega)}$ , result in the so called “bounded variation regularization methods” ([1], [12]). The use of these penalizers is appropriate when preserving discontinuities or edges is an important matter. The method, however, has as a drawback that it tends to produce piecewise constant approximations and therefore, it will most likely be inappropriate in regions where the exact solution is smooth ([5]) producing the so called “staircasing effect”.

In certain types of problems, particularly in those in which it is known that the regularity of the exact solution is heterogeneous and/or anisotropic, it is reasonable to think that using and spatially adapting two or more penalizers of different nature could be more convenient. During the last 15 years several regularization methods have been developed in light of this reasoning. Thus, for instance, in 1997 Blomgren *et al.* ([4]) proposed the use of the following penalizer, by using variable  $L^p$  spaces:

$$W(u) = \int_{\Omega} |\nabla u|^{p(|\nabla u|)} dx, \quad (3)$$

where  $\lim_{u \rightarrow 0^+} p(u) = 2$ ,  $\lim_{u \rightarrow \infty} p(u) = 1$  and  $p$  is a decreasing function. Thus, in regions where the modulus of the gradient of  $u$  is small the penalizer is approximately equal to  $\|\nabla u\|_{L^2(\Omega)}^2$  corresponding to a Tikhonov-Phillips method of order one (appropriate for restoration in smooth regions). On the other hand, when the modulus of the gradient of  $u$  is large, the penalizer resembles the bounded variation seminorm  $\|\nabla u\|_{L^1(\Omega)}$ , whose use, as mentioned earlier, is highly appropriate for border detection purposes. Although this model for  $W$  is quite reasonable, proving basic properties of the corresponding generalized Tikhonov-Phillips functional turns out to be quite difficult. A different way of combining these two methods was proposed by Chambolle and Lions ([5]). They suggested the use of a thresholded penalizer of the form:

$$W_\beta(u) = \int_{|\nabla u| \leq \beta} |\nabla u|^2 dx + \int_{|\nabla u| > \beta} |\nabla u| dx,$$

where  $\beta > 0$  is a prescribed threshold parameter. Thus, in regions where borders are more likely to be present ( $|\nabla u| > \beta$ ), penalization is made with the bounded variation seminorm

while a standard order-one Tikhonov-Phillips method is used otherwise. This model was shown to be successful in restoring images possessing regions with homogeneous intensity separated by borders. However, in the case of images with non-uniform or highly degraded intensities, the model is extremely sensitive to the choice of the threshold parameter  $\beta$ . More recently, penalizers of the form

$$W(u) = \int_{\Omega} |\nabla u|^{p(x)} dx, \quad (4)$$

for certain functions  $p$  with range in  $[1, 2]$ , were studied in [6] and [10]. It is timely to point out here that all previously mentioned results work only for the case of denoising, i.e. for the case  $T = id$ .

In this work we propose the use of a model for general restoration problems, which combines, in an appropriate way, the penalizers corresponding to a zero-order Tikhonov-Phillips method and the bounded variation seminorm. Although several mathematical issues for this model still remain open, its use in some signal and image restoration problems has already proved to be very promising. The purpose of this article is to introduce the model, show mathematical results regarding the existence of the corresponding regularized solutions, and present some results of its application to signal restoration.

The following Theorem, whose proof can be found in [1] (Theorem 3.1), guarantees the well-posedness of the unconstrained minimization problem

$$u^* = \operatorname{argmin}_{u \in L^p(\Omega)} J(u). \quad (5)$$

**Theorem 1.1.** *Let  $J$  be a BV-coercive functional defined on  $L^p(\Omega)$ . If  $1 \leq p < \frac{n}{n-1}$  and  $J$  is lower semicontinuous, then problem (5) has a solution. If  $p = \frac{n}{n-1}$ ,  $n \geq 2$  and in addition  $J$  is weakly lower semicontinuous, then a solutions also exists. In either case, the solution is unique if  $J$  is strictly convex.*

The following theorem, whose proof can also be found in [1] (Theorem 4.1), is very important for the existence and uniqueness of minimizers of functionals of the form

$$J(u) = \|Tu - v\|^2 + \alpha J_0(u), \quad (6)$$

where  $\alpha > 0$  and  $J_0(u)$  denotes the bounded variation seminorm given by

$$J_0(u) = \sup_{\vec{\nu} \in \mathcal{V}} \int_{\Omega} -u \operatorname{div} \vec{\nu} dx, \quad (7)$$

with  $\mathcal{V} \doteq \{\vec{\nu} : \Omega \rightarrow \mathbb{R}^n \text{ such that } \vec{\nu} \in C_0^1(\Omega) \text{ and } |\vec{\nu}(x)| \leq 1 \forall x \in \Omega\}$ .

**Theorem 1.2.** *Suppose that  $p$  satisfies the restrictions of Theorem 1.1 and  $T\chi_{\Omega} \neq 0$ . Then  $J(\cdot)$  defined by (6) is BV-coercive.*

Note here that (6) is a particular case of (2) with  $W(u) = J_0(u)$ . The following theorem, whose proof can be found in [11], gives conditions guaranteeing existence and uniqueness of minimizers of (2) for general penalizers  $W(u)$ . This theorem will also be very important for our main results in the next section.

**Theorem 1.3.** *Let  $\mathcal{X}, \mathcal{Y}$  be normed vector spaces,  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $v \in \mathcal{Y}$ ,  $\mathcal{D} \subset \mathcal{X}$  a convex set and  $W : \mathcal{D} \rightarrow \mathbb{R}$  a functional bounded from below,  $W$ -subsequentially weakly lower semicontinuous, and such that  $W$ -bounded sets are relatively weakly compact in  $\mathcal{X}$ . More precisely, suppose that  $W$  satisfies the following hypotheses:*

- (H1):  $\exists \gamma \geq 0$  such that  $W(u) \geq -\gamma \quad \forall u \in \mathcal{D}$ .
- (H2): for every  $W$ -bounded sequence  $\{u_n\} \subset \mathcal{D}$  such that  $u_n \xrightarrow{w} u \in \mathcal{D}$ , there exists a subsequence  $\{u_{n_j}\} \subset \{u_n\}$  such that  $W(u) \leq \liminf_{j \rightarrow \infty} W(u_{n_j})$ .
- (H3): for every  $W$ -bounded sequence  $\{u_n\} \subset \mathcal{D}$  there exist a subsequence  $\{u_{n_j}\} \subset \{u_n\}$  and  $u \in \mathcal{D}$  such that  $u_{n_j} \xrightarrow{w} u$ .

Then the functional  $J_{W,\alpha}(u) \doteq \|Tu - v\|^2 + \alpha W(u)$  has a global minimizer on  $\mathcal{D}$ . If moreover  $W$  is convex and  $T$  is injective or if  $W$  is strictly convex, then such a minimizer is unique.

*Proof.* See Theorem 2.5 in [11]. □

## 2 Main results

In this section we will state and prove our main results concerning existence and uniqueness of minimizers of particular generalized Tikhonov-Phillips functionals with combined spatially-varying  $L^2$ -BV penalizers. In what follows  $\mathcal{M}(\Omega)$  shall denote the set of all real valued measurable functions defined on  $\Omega$  and  $\widehat{\mathcal{M}}(\Omega)$  the subset of  $\mathcal{M}(\Omega)$  formed by those functions with values in  $[0, 1]$ .

**Definition 2.1.** Given  $\theta \in \widehat{\mathcal{M}}(\Omega)$  we define the functional  $W_{0,\theta}(u)$  with values on the extended reals by

$$W_{0,\theta}(u) \doteq \sup_{\vec{v} \in \mathcal{V}_\theta} \int_{\Omega} -u \operatorname{div}(\theta \vec{v}) \, dx, \quad u \in \mathcal{M}(\Omega) \quad (8)$$

where  $\mathcal{V}_\theta \doteq \{\vec{v} : \Omega \rightarrow \mathbb{R}^n \text{ such that } \theta \vec{v} \in C_0^1(\Omega) \text{ and } |\vec{v}(x)| \leq 1 \, \forall x \in \Omega\}$ .

**Lemma 2.2.** If  $u \in C^1(\Omega)$  then  $W_{0,\theta}(u) = \|\theta |\nabla u|\|_{L^1(\Omega)}$ .

*Proof.* Let  $u \in C^1(\Omega)$ . For all  $\vec{v} \in \mathcal{V}_\theta$  it follows easily that

$$\begin{aligned} \int_{\Omega} -u \operatorname{div}(\theta \vec{v}) \, dx &= \int_{\Omega} \nabla u \cdot \theta \vec{v} \, dx - \int_{\partial\Omega} (u \theta \vec{v} \cdot \vec{n}) \, dS \\ &= \int_{\Omega} \nabla u \cdot \theta \vec{v} \, dx \quad (\text{since } \theta \vec{v}|_{\partial\Omega} = 0) \\ &\leq \int_{\Omega} |\theta \nabla u| |\vec{v}| \, dx \\ &\leq \int_{\Omega} |\theta \nabla u| \, dx \quad (\text{since } |\vec{v}(x)| \leq 1), \end{aligned} \quad (9)$$

where  $\vec{n}$  denotes the outward unit normal to  $\partial\Omega$ . Taking supremum over  $\vec{v} \in \mathcal{V}_\theta$  it follows that

$$W_{0,\theta}(u) \leq \|\theta |\nabla u|\|_{L^1(\Omega)}.$$

For the opposite inequality, define  $\vec{v}_*(x) \doteq \begin{cases} \frac{\nabla u(x)}{|\nabla u(x)|}, & \text{if } |\nabla u(x)| \neq 0, \\ 0, & \text{if } |\nabla u(x)| = 0, \end{cases}$ . Then one has that  $|\vec{v}_*(x)| \leq 1 \, \forall x \in \Omega$  and  $\vec{v}_* \in C(\Omega, \mathbb{R}^n)$  since  $u \in C^1(\Omega)$ . Also,

$$\int_{\Omega} (\nabla u \cdot \theta \vec{v}_*) \, dx = \int_{\Omega} |\theta \nabla u| \, dx.$$

By convolving  $\vec{\nu}_*$  with an appropriately chosen function  $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$ , we can obtain a function  $\vec{\nu} \in \mathcal{V}_\theta \cap C_0^\infty(\Omega, \mathbb{R}^n)$  for which the left hand side of (9) is arbitrarily close to  $\int_\Omega |\theta \nabla u| \, dx$ . Then taking supremum over  $\vec{\nu} \in \mathcal{V}_\theta$  we have that

$$W_{0,\theta}(u) \geq \|\theta |\nabla u|\|_{L^1(\Omega)}.$$

Hence  $W_{0,\theta}(u) = \|\theta |\nabla u|\|_{L^1(\Omega)}$ , as we wanted to prove.  $\square$

Observation: From the density of  $C^1(\Omega)$  in  $W^{1,1}(\Omega)$  it follows that Lemma 2.2 holds for every  $u \in W^{1,1}(\Omega)$ .

**Remark 2.3.** For any  $\theta \in \widehat{\mathcal{M}}(\Omega)$ , it follows easily that

$$W_{0,\theta}(u) \leq J_0(u), \quad \forall u \in \mathcal{M}(\Omega). \quad (10)$$

In fact, for any  $\vec{\nu} \in \mathcal{V}_\theta$  and for any  $u \in \mathcal{M}(\Omega)$  we have that

$$\begin{aligned} \int_\Omega -u \operatorname{div}(\theta \vec{\nu}) \, dx &\leq \sup_{\vec{\nu} \in \mathcal{V}} \int_\Omega -u \operatorname{div} \vec{\nu} \, dx \\ &= J_0(u), \end{aligned} \quad (11)$$

where inequality (11) follows from the fact that  $\theta \vec{\nu} \in \mathcal{V}$  (since  $|\theta(x)| \leq 1 \, \forall x \in \Omega$ ). By taking supremum for  $\vec{\nu} \in \mathcal{V}_\theta$  inequality (10) follows.

Although inequality (10) is important by itself since it relates the functionals  $W_{0,\theta}$  and  $J_0$ , in order to be able to use the known coercitivity properties of  $J_0$  (see [1]), an inequality of the opposite type is highly desired. That is, we would like to show that, under certain conditions on  $\theta(\cdot)$ , there exists a constant  $C = C(\theta)$  such that  $W_{0,\theta}(u) \geq C J_0(u)$  for all  $u \in \mathcal{M}(\Omega)$ . The following theorem provides sufficient conditions on  $\theta$  assuring such an inequality.

**Theorem 2.4.** Let  $\theta \in \widehat{\mathcal{M}}(\Omega)$  be such that  $\frac{1}{\theta} \in L^\infty(\Omega)$  and let  $J_0, W_{0,\theta}$  be the functionals defined in (7) and (8), respectively. Then  $J_0(u) \leq \|\frac{1}{\theta}\|_{L^\infty(\Omega)} W_{0,\theta}(u)$  for all  $u \in \mathcal{M}(\Omega)$ .

*Proof.* Let  $u \in \mathcal{M}(\Omega)$  and  $K_\theta \doteq \|\frac{1}{\theta}\|_{L^\infty(\Omega)}$ . Then for all  $\vec{\nu} \in \mathcal{V}$

$$\begin{aligned} \int_\Omega -u \operatorname{div} \vec{\nu} \, dx &= K_\theta \int_\Omega -u \operatorname{div} \left( \frac{\theta \vec{\nu}}{K_\theta \theta} \right) \, dx \\ &\leq K_\theta \sup_{\vec{\omega} \in \mathcal{V}_\theta} \int_\Omega -u \operatorname{div} (\theta \vec{\omega}) \, dx \\ &= K_\theta W_{0,\theta}(u), \end{aligned}$$

where the last inequality follows from the fact that  $\frac{\vec{\nu}}{K_\theta \theta} \in \mathcal{V}_\theta$  since  $K_\theta \geq 1, |K_\theta \theta(x)| \geq 1 \, \forall x \in \Omega$  and  $\vec{\nu} \in \mathcal{V}$ . Then, taking supremum for  $\vec{\nu} \in \mathcal{V}$  we conclude that  $J_0(u) \leq K_\theta W_{0,\theta}(u)$ .  $\square$

The following lemma will be of fundamental importance for proving several of the upcoming results.

**Lemma 2.5.** The functional  $W_{0,\theta}$  defined by (8) is weakly lower semicontinuous with respect to the  $L^p$  topology,  $\forall p \in [1, \infty)$ .

*Proof.* Let  $p \in [1, \infty)$ ,  $\{u_n\} \subset L^p(\Omega)$  and  $u \in L^p(\Omega)$  be such that  $u_n \xrightarrow{w} u$ . Let  $\vec{v}_* \in \mathcal{V}_\theta$  and  $q$  the conjugate dual of  $p$ . Since  $\theta \vec{v}_* \in C_0^1(\Omega)$ , it follows that  $\operatorname{div}(\theta \vec{v}_*)$  is uniformly bounded on  $\Omega$  and therefore,  $\operatorname{div}(\theta \vec{v}_*) \in L^\infty(\Omega) \subset L^q(\Omega)$ . Then, from the weak convergence of  $u_n$  it follows that  $\lim_{n \rightarrow \infty} \int_\Omega -u_n \operatorname{div}(\theta \vec{v}_*) dx = \int_\Omega -u \operatorname{div}(\theta \vec{v}_*) dx$ .

Hence  $\int_\Omega -u \operatorname{div}(\theta \vec{v}_*) dx = \lim_{n \rightarrow \infty} \int_\Omega -u_n \operatorname{div}(\theta \vec{v}_*) dx \leq \liminf_{n \rightarrow \infty} \sup_{\vec{v} \in \mathcal{V}_\theta} \int_\Omega -u_n \operatorname{div}(\theta \vec{v}) dx = \liminf_{n \rightarrow \infty} W_{0,\theta}(u_n)$ . Thus  $\forall \vec{v}_* \in \mathcal{V}_\theta$

$$\int_\Omega -u \operatorname{div}(\theta \vec{v}_*) dx \leq \liminf_{n \rightarrow \infty} W_{0,\theta}(u_n).$$

Taking supremum over all  $\vec{v}_* \in \mathcal{V}_\theta$  it follows that  $W_{0,\theta}(u) \leq \liminf_{n \rightarrow \infty} W_{0,\theta}(u_n)$ .  $\square$

We are now ready to present several results on existence and uniqueness of minimizers of generalized Tikhonov-Phillips functionals with penalizers involving spatially varying combinations of the  $L^2$ -norm and of the functional  $W_{0,\theta}$ , under different hypotheses on the function  $\theta$ .

**Theorem 2.6.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open convex set with Lipschitz boundary,  $\mathcal{X} = L^2(\Omega)$ ,  $\mathcal{Y}$  a normed vector space,  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $v \in \mathcal{Y}$ ,  $\alpha_1, \alpha_2$  positive constants,  $\theta \in \widehat{\mathcal{M}}(\Omega)$  and  $F_\theta$  the functional defined by*

$$F_\theta(u) \doteq \|Tu - v\|_{\mathcal{Y}}^2 + \alpha_1 \|\sqrt{1 - \theta} u\|_{L^2(\Omega)}^2 + \alpha_2 W_{0,\theta}(u), \quad u \in \mathcal{D} \doteq L^2(\Omega). \quad (12)$$

*If there exists  $\varepsilon_2 \in \mathbb{R}$ , such that  $\theta(x) \leq \varepsilon_2 < 1$  for a.e.  $x \in \Omega$ , then the functional (12) has a unique global minimizer  $u^* \in L^2(\Omega)$ . If moreover there exists  $\varepsilon_1 \in \mathbb{R}$  such that  $0 < \varepsilon_1 \leq \theta(x)$  for a.e.  $x \in \Omega$ , then  $u^* \in BV(\Omega)$ .*

*Proof.* By virtue of Theorem 1.3 it is sufficient to show that the functional

$$W(u) \doteq \alpha_1 \|\sqrt{1 - \theta} u\|_{L^2(\Omega)}^2 + \alpha_2 W_{0,\theta}(u), \quad u \in L^2(\Omega)$$

satisfies hypotheses (H1), (H2) and (H3). Clearly (H1) holds with  $\gamma = 0$ .

To prove (H2) let  $\{u_n\} \subset L^2(\Omega)$  such that  $u_n \xrightarrow{w} u \in L^2(\Omega)$  and  $W(u_n) \leq c_1 < \infty$ . We want to show that  $W(u) \leq \liminf_{n \rightarrow \infty} W(u_n)$ . Since  $\sqrt{1 - \theta} \in L^\infty(\Omega)$  one has  $\sqrt{1 - \theta} u_n \xrightarrow{w} \sqrt{1 - \theta} u$ .

The condition  $\theta(x) \leq \varepsilon_2 < 1$  for a.e.  $x \in \Omega$ , clearly implies that  $\|\sqrt{1 - \theta} \cdot\|_{L^2(\Omega)}$  is a norm. Then, from the weak lower semicontinuity of  $\|\sqrt{1 - \theta} \cdot\|_{L^2(\Omega)}^2$ , it follows that

$$\|\sqrt{1 - \theta} u\|_{L^2(\Omega)}^2 \leq \liminf_{n \rightarrow \infty} \|\sqrt{1 - \theta} u_n\|_{L^2(\Omega)}^2. \quad (13)$$

On the other hand, from the weak lower semicontinuity of  $W_{0,\theta}$  in  $L^2(\Omega)$  (see Lemma 2.5) it follows that

$$W_{0,\theta}(u) \leq \liminf_{n \rightarrow \infty} W_{0,\theta}(u_n). \quad (14)$$

From (13) and (14) we then conclude that

$$\begin{aligned}
W(u) &= \alpha_1 \|\sqrt{1-\theta} u\|_{L^2(\Omega)}^2 + \alpha_2 W_{0,\theta}(u) \\
&\leq \alpha_1 \liminf_{n \rightarrow \infty} \|\sqrt{1-\theta} u_n\|_{L^2(\Omega)}^2 + \alpha_2 \liminf_{n \rightarrow \infty} W_{0,\theta}(u_n) \\
&\leq \liminf_{n \rightarrow \infty} \left( \alpha_1 \|\sqrt{1-\theta} u_n\|_{L^2(\Omega)}^2 + \alpha_2 W_{0,\theta}(u_n) \right) \\
&= \liminf_{n \rightarrow \infty} W(u_n),
\end{aligned}$$

what proves (H2).

To prove (H3) let  $\{u_n\} \subset L^2(\Omega)$  be such that  $W(u_n) \leq c_1 < \infty$ ,  $\forall n$ . We want to show that there exist  $\{u_{n_j}\} \subset \{u_n\}$  and  $u \in L^2(\Omega)$  such that  $u_{n_j} \xrightarrow{w} u$ . For this note that

$$(1 - \varepsilon_2) \|u_n\|_{L^2(\Omega)}^2 \leq \|\sqrt{1-\theta} u_n\|_{L^2(\Omega)}^2 \leq W(u_n) \leq c_1. \quad (15)$$

Thus  $\|u_n\|_{L^2(\Omega)}$  is uniformly bounded and therefore there exist  $\{u_{n_j}\} \subset \{u_n\}$  and  $u^* \in L^2(\Omega)$  such that  $u_{n_j} \xrightarrow{w} u^*$ . Hence, by Theorem 1.3, the functional  $F_\theta(u)$  given by (12) has a global minimizer  $u^* \in L^2(\Omega)$ . The condition  $\theta(x) \leq \varepsilon_2 < 1$  for a.e.  $x \in \Omega$  clearly implies the strict convexity of  $F_\theta$  and therefore the uniqueness of such a global minimizer.

To prove the second part of the theorem, assume further that there exists  $\varepsilon_1 > 0$  such that  $\theta(x) \geq \varepsilon_1$  for a.e.  $x \in \Omega$ . Following the proof of Theorem 5.1 in [11], it suffices to show that under this additional hypothesis the weak limit  $u$  in (H3) above belongs to  $BV(\Omega)$ . For this note that from (15) it follows that there exist  $c_2 < \infty$  such that

$$\|u_n\|_{L^1(\Omega)} \leq c_2 \quad \forall n. \quad (16)$$

Also, by Theorem 2.4  $W_{0,\theta}(u) \geq \varepsilon_1 J_0(u) \quad \forall u \in \mathcal{M}(\Omega)$ . This, together with (16) implies that

$$\|u_n\|_{BV(\Omega)} = \|u_n\|_{L^1(\Omega)} + J_0(u_n) \leq c_2 + \frac{W_{0,\theta}(u_n)}{\varepsilon_1} \leq c_3 < \infty \quad \forall n,$$

where the previous to last inequality follows from the uniform boundedness of  $W_{0,\theta}(u_n)$ , which, in turn, follows from the uniform boundedness of  $W(u_n)$ . Hence the fact that the weak limit in (H3) is in  $BV(\Omega)$  follows from the compact embedding of  $BV(\Omega)$  in  $L^2(\Omega)$ . This result is an extension of the Rellich-Kondrachov Theorem which can be found, for instance, in [2] and [3].  $\square$

**Remark 2.7.** Note that if  $\theta(x) = 0 \quad \forall x \in \Omega$ , then  $W(u) = \|u\|_{L^2(\Omega)}^2$  and  $F_\theta$  as defined in (12) is the classical Tikhonov-Phillips functional of order zero. On the other hand, if  $\theta(x) = 1 \quad \forall x \in \Omega$  then  $W(u) = J_0(u)$  and  $F_\theta$  has a global minimizer provided that  $T\chi_\Omega \neq 0$ . If moreover  $T$  is injective then such a global minimizer is unique. All these facts follow immediately from Theorems 3.1 and 4.1 in [1].

**Theorem 2.8.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded open convex set with Lipschitz boundary,  $\mathcal{X} = L^2(\Omega)$ ,  $\mathcal{Y}$  a normed vector space,  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $v \in \mathcal{Y}$ ,  $\alpha_1, \alpha_2$  positive constants and  $\theta \in \widehat{\mathcal{M}}(\Omega)$  such that  $\frac{1}{1-\theta} \in L^1(\Omega)$  and  $\frac{1}{\theta} \in L^\infty(\Omega)$ . Then the functional (12) has a unique global minimizer  $u^* \in BV(\Omega)$ .



*Proof.* Let us consider the functional

$$W(u) \doteq \alpha_1 \|\sqrt{1-\theta} u\|_{L^2(\Omega)}^2 + \alpha_2 W_{0,\theta}(u), \quad u \in L^2(\Omega).$$

By virtue of Theorems 1.3 and 2.6 and the compact embedding of  $BV(\Omega)$  in  $L^2(\Omega)$ , it suffices to show that  $W(\cdot)$  satisfies (H1) and (H2) and that every  $W$ -bounded sequence is also  $BV$ -bounded. Clearly  $W(\cdot)$  satisfies (H1) with  $\gamma = 0$ . That it satisfies (H2) follows immediately from the fact that the condition  $\frac{1}{1-\theta} \in L^1(\Omega)$  implies that  $\|\sqrt{1-\theta} \cdot\|_{L^2(\Omega)}$  is a norm.

Now, let  $\{u_n\} \subset L^2(\Omega)$  be a  $W$ -bounded sequence, i.e. such that  $W(u_n) \leq c < \infty$ ,  $\forall n$ . We will show that  $\{u_n\}$  is  $BV$ -bounded. Since  $W(u_n)$  is uniformly bounded, there exist  $K < \infty$  such that  $\|\sqrt{1-\theta} u_n\|_{L^2(\Omega)} \leq K \forall n$ . From this and the fact that  $\frac{1}{1-\theta} \in L^1(\Omega)$  it follows that

$$\begin{aligned} \|u_n\|_{L^1(\Omega)} &= \int_{\Omega} \frac{1}{\sqrt{1-\theta}} \sqrt{1-\theta} |u_n| dx \\ &\leq \left( \int_{\Omega} \frac{1}{1-\theta} dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (1-\theta) u_n^2 dx \right)^{\frac{1}{2}} \\ &= \left\| \frac{1}{1-\theta} \right\|_{L^1(\Omega)}^{\frac{1}{2}} \|\sqrt{1-\theta} u_n\|_{L^2(\Omega)} \\ &\leq K \left\| \frac{1}{1-\theta} \right\|_{L^1(\Omega)}^{\frac{1}{2}} < \infty \quad \forall n. \end{aligned} \tag{17}$$

On the other hand from Theorem 2.4  $J_0(u) \leq W_{0,\theta}(u) \left\| \frac{1}{\theta} \right\|_{L^\infty(\Omega)} \forall u \in L^2(\Omega)$ . Since  $\frac{1}{\theta} \in L^\infty(\Omega)$  and  $W_{0,\theta}(u_n)$  is uniformly bounded, it then follows that there exists  $C < \infty$  such that

$$J_0(u_n) \leq C \forall n. \tag{18}$$

From (17) and (18) it follows that

$$\|u_n\|_{BV(\Omega)} = \|u_n\|_{L^1(\Omega)} + J_0(u_n) \leq K \left\| \frac{1}{1-\theta} \right\|_{L^1(\Omega)}^{\frac{1}{2}} + C < \infty \forall n.$$

Hence  $\{u_n\}$  is  $BV$ -bounded. The existence of a global minimizer of functional (12) belonging to  $BV(\Omega)$  then follows. Finally note that the condition  $\frac{1}{1-\theta} \in L^1(\Omega)$  implies the strict convexity of  $F_\theta$  and therefore the uniqueness of the global minimizer.  $\square$

**Remark 2.9.** Note that the condition  $\frac{1}{1-\theta} \in L^1(\Omega)$  in Theorem 2.8 is weaker than the condition  $\theta(x) \leq \varepsilon_2 < 1$  for a.e.  $x \in \Omega$  of Theorem 2.6. While the latter suffices to guarantee the existence of a global minimizer in  $L^2(\Omega)$ , the former does not. However this weaker condition  $\frac{1}{1-\theta} \in L^1(\Omega)$  together with the condition  $\frac{1}{\theta} \in L^\infty(\Omega)$  are enough for guaranteeing not only the existence of a unique global minimizer, but also the fact that such a minimizer belongs to  $BV(\Omega)$ .

It is timely to note that in both Theorems 2.6 and 2.8, the function  $\theta$  cannot assume the extreme values 0 and 1 on a set of positive measure. In some cases a pure  $BV$  regularization in some regions and a pure  $L^2$  regularization in others may be desired, and therefore that restraint on the function  $\theta$  will turn out to be inappropriate. In the next three theorems we introduce different conditions which allow the function  $\theta$  to take the extreme values on sets of positive measure.



**Theorem 2.10.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open convex set with Lipschitz boundary,  $\mathcal{X} = L^2(\Omega)$ ,  $\mathcal{Y}$  a normed vector space,  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $v \in \mathcal{Y}$ ,  $\alpha_1, \alpha_2$  positive constants,  $\theta \in \widehat{\mathcal{M}}(\Omega)$  and  $\Omega_0 \doteq \{x \in \Omega \text{ such that } \theta(x) = 0\}$ . If  $\frac{1}{\theta} \in L^\infty(\Omega_0^c)$  and  $\frac{1}{1-\theta} \in L^1(\Omega_0^c)$  then functional (12) has a unique global minimizer  $u^* \in L^2(\Omega) \cap BV(\Omega_0^c)$ .*

*Proof.* Under the hypotheses of the theorem the functional  $W(u)$  can be written as

$$W(u) = \alpha_1 \|u\|_{L^2(\Omega_0)}^2 + \alpha_1 \|\sqrt{1-\theta} u\|_{L^2(\Omega_0^c)}^2 + \alpha_2 \sup_{\vec{\nu} \in \mathcal{V}_\theta} \int_{\Omega_0^c} -u|_{\Omega_0^c} \operatorname{div}(\theta \vec{\nu}) dx. \quad (19)$$

Just like in Theorem 2.8 it follows easily that  $W(\cdot)$  satisfies (H1) and (H2).

Let now  $\{u_n\} \subset L^2(\Omega)$  be a  $W$ -bounded sequence. From (19) we conclude that there exist  $u_1^* \in L^2(\Omega_0)$  and a subsequence  $\{u_{n_j}\} \subset \{u_n\}$  such that  $u_{n_j}|_{\Omega_0} \xrightarrow{w-L^2(\Omega_0)} u_1^*$ . On the other hand from the uniform boundedness of  $\sup_{\vec{\nu} \in \mathcal{V}_\theta} \int_{\Omega_0^c} -u_{n_j}|_{\Omega_0^c} \operatorname{div}(\theta \vec{\nu}) dx$ , by using Theorem 2.4 with  $\Omega$  replaced by  $\Omega_0^c$ , it follows that there exists a constant  $C \leq \infty$  such that  $J_0(u_{n_j}|_{\Omega_0^c}) \leq C$  for all  $n_j$ . Also, from (19) and the hypothesis that  $\frac{1}{1-\theta} \in L^1(\Omega_0^c)$ , it can be easily proved that the sequence  $\{u_n\}$  is uniformly bounded in  $L^1(\Omega_0^c)$ . Hence  $\{u_{n_j}|_{\Omega_0^c}\}$  is uniformly  $BV$ -bounded. By using the compact embedding of  $BV(\Omega_0^c)$  in  $L^2(\Omega_0^c)$  it follows that there exist a subsequence  $\{u_{n_{j_k}}\}$  of  $\{u_{n_j}\}$  and  $u_2^* \in BV(\Omega_0^c)$  such that  $u_{n_{j_k}} \xrightarrow{w-L^2(\Omega_0^c)} u_2^*$ .

Let us define now

$$\begin{aligned} \hat{u}_1(x) &\doteq \begin{cases} u_1^*(x), & \text{if } x \in \Omega_0, \\ 0, & \text{if } x \in \Omega_0^c, \end{cases} \\ \hat{u}_2(x) &\doteq \begin{cases} u_2^*(x), & \text{if } x \in \Omega_0^c, \\ 0, & \text{if } x \in \Omega_0, \end{cases} \end{aligned}$$

and  $u^* \doteq \hat{u}_1 + \hat{u}_2$ . Then one has that  $u^* \in L^2(\Omega)$ ,  $u^*|_{\Omega_0^c} = u_2^* \in BV(\Omega_0^c)$  and  $u_{n_{j_k}} \xrightarrow{w-L^2(\Omega)} u^*$ .

The existence of a global minimizer of functional (12) then follows immediately from Theorem 1.3. Uniqueness is a consequence of the fact that the hypothesis  $\frac{1}{1-\theta} \in L^1(\Omega_0^c)$  implies that  $\|\sqrt{1-\theta} \cdot\|_{L^2(\Omega_0^c)}$  is a norm.  $\square$

**Theorem 2.11.** *Let  $n \leq 2$ ,  $\Omega \subset \mathbb{R}^n$  be a bounded open convex set with Lipschitz boundary,  $\mathcal{X} = L^2(\Omega)$ ,  $\mathcal{Y}$  a normed vector space,  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $v \in \mathcal{Y}$ ,  $\alpha_1, \alpha_2$  positive constants. Let  $\theta \in \widehat{\mathcal{M}}(\Omega)$  and  $\Omega_1 \doteq \{x \in \Omega \text{ such that } \theta(x) = 1\}$ . If  $\frac{1}{\theta} \in L^\infty(\Omega_1^c)$ ,  $\frac{1}{1-\theta} \in L^1(\Omega_1^c)$  and  $T\chi_{\Omega} \neq 0$ , then the functional (12) has a global minimizer  $u^* \in L^2(\Omega) \cap BV(\Omega_1^c)$ . If moreover  $\mathcal{N}(T)$  does not contain functions vanishing on  $\Omega_1$ , i.e. if  $Tu = 0$  implies  $u|_{\Omega_1} \neq 0$ , then such a global minimizer is unique.*

*Proof.* We will prove that under the hypotheses of the theorem, the functional  $F_\theta(\cdot)$  defined by (12) is weakly lower semicontinuous with respect to the  $L^2(\Omega)$  topology and  $BV$ -coercive.

First note that under the hypotheses of the theorem we can write

$$F_\theta(u) = \|Tu - v\|_{\mathcal{Y}}^2 + \alpha_1 \|\sqrt{1-\theta} u\|_{L^2(\Omega_1^c)}^2 + \alpha_2 W_{0,\theta}(u). \quad (20)$$

Since  $\frac{1}{1-\theta} \in L^1(\Omega_1^c)$ , it follows that  $\|\sqrt{1-\theta} \cdot\|_{L^2(\Omega_1^c)}$  is a norm in  $L^2(\Omega_1^c)$  and therefore it is weakly lower semicontinuous. The weak lower semicontinuity of  $F_\theta(\cdot)$  then follows immediately from this fact, from Lemma 2.5 and from the weak lower semicontinuity of the norm in  $\mathcal{Y}$ .

For the  $BV$ -coercitivity, note that

$$\begin{aligned}
\|Tu - v\|^2 + \alpha_2 J_0(u) &\leq \|Tu - v\|^2 + \alpha_2 \left\| \frac{1}{\theta} \right\|_{L^\infty(\Omega_1^c)} W_{0,\theta}(u) \quad (\text{from Theorem (2.4)}) \\
&\leq \|Tu - v\|^2 + \alpha_2 \left\| \frac{1}{\theta} \right\|_{L^\infty(\Omega_1^c)} W_{0,\theta}(u) + \alpha_1 \|\sqrt{1-\theta} u\|_{L^2(\Omega_1^c)}^2 \\
&\leq \left\| \frac{1}{\theta} \right\|_{L^\infty(\Omega_1^c)} F_\theta(u) \quad (\text{since } \|\theta^{-1}\|_{L^\infty(\Omega_1^c)} \geq 1). \tag{21}
\end{aligned}$$

Now, since  $T\chi_\Omega \neq 0$ , by Theorem 1.2 the functional  $J(u) \doteq \|Tu - v\|^2 + \alpha_2 J_0(u)$  is  $BV$ -coercive on  $L^2(\Omega)$ . From this and inequality (21) it follows that  $F_\theta(\cdot)$  is also  $BV$ -coercive. The existence of a global minimizer  $u^* \in L^2(\Omega)$  then follows from Theorem 1.1. Since  $F_\theta(u^*) < \infty$  it follows that both  $\|u^*\|_{L^1(\Omega_1^c)}$  and  $W_{0,\theta}(u^*)$  are finite. The fact that  $u^*$  is of bounded variation on  $\Omega_1^c$  then follows from Theorem 2.4. Finally, if  $\mathcal{N}(T)$  does not contain functions vanishing on  $\Omega_1$  then it follows easily that  $F_\theta(u)$  is strictly convex and therefore such a global minimizer is unique.  $\square$

**Theorem 2.12.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \leq 2$  be a bounded open convex set with Lipschitz boundary,  $\mathcal{X} = L^2(\Omega)$ ,  $\mathcal{Y}$  a normed vector space,  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $v \in \mathcal{Y}$ ,  $\alpha_1, \alpha_2$  positive constants. Let  $\theta \in \widehat{\mathcal{M}}(\Omega)$ ,  $\Omega_0 \doteq \{x \in \Omega \text{ such that } \theta(x) = 0\}$  and  $\Omega_1 \doteq \{x \in \Omega \text{ such that } \theta(x) = 1\}$ . If  $\frac{1}{\theta} \in L^\infty(\Omega_0^c)$ ,  $\frac{1}{1-\theta} \in L^\infty(\Omega_1^c)$  and  $\mathcal{N}(T)$  does not contain functions vanishing on  $\Omega_1$ , i.e. if  $Tu = 0$  implies  $u|_{\Omega_1} \neq 0$ , then functional (12) has a unique global minimizer  $u^* \in L^2(\Omega) \cap BV(\Omega_1^c \cap \Omega_0^c)$ .*

*Proof.* For the existence of a global minimizer it is sufficient to prove that the functional  $F_\theta$  defined by (20) is weakly lower semicontinuous and  $L^2(\Omega)$ -coercive. For this, note that

$$F_\theta(u) = \|Tu - v\|_{\mathcal{Y}}^2 + \alpha_1 \|\sqrt{1-\theta} u\|_{L^2(\Omega_1^c)}^2 + \alpha_2 \sup_{\vec{v} \in \mathcal{V}_\theta} \int_{\Omega_0^c} -u \operatorname{div}(\theta \vec{v}) dx. \tag{22}$$

The weak lower semicontinuity of  $F_\theta(\cdot)$  follows from Lemma 2.5 and the weak lower semicontinuity of the norms in  $\mathcal{Y}$  and  $\|\sqrt{1-\theta} \cdot\|_{L^2(\Omega_1^c)}$ .

We shall now prove that  $F_\theta(\cdot)$  is  $L^2(\Omega)$ -coercive. For that, assume  $\{u_n\}$  is a sequence in  $L^2(\Omega)$  such that  $\|u_n\|_{L^2(\Omega)} \rightarrow \infty$ . Then either  $\|u_n\|_{L^2(\Omega_1^c)} \rightarrow \infty$  or  $\|u_n\|_{L^2(\Omega_1)} \rightarrow \infty$ . If  $\|u_n\|_{L^2(\Omega_1^c)} \rightarrow \infty$ , then the hypothesis  $\frac{1}{1-\theta} \in L^\infty(\Omega_1^c)$  implies that  $\|\sqrt{1-\theta} u\|_{L^2(\Omega_1^c)}^2 \rightarrow \infty$  and therefore  $F_\theta(u_n) \rightarrow \infty$ . Suppose now that  $\|u_n\|_{L^2(\Omega_1)} \rightarrow \infty$  and without loss of generality assume that  $\|u_n\|_{L^2(\Omega_1^c)} \leq C < \infty$ . Then due to the compact embedding  $BV(\Omega_1) \hookrightarrow L^2(\Omega_1)$  it follows that  $\|u_n\|_{BV(\Omega_1)} \rightarrow \infty$ . Since  $\mathcal{N}(T)$  does not contain functions vanishing on  $\Omega_1$ , it follows that  $T\chi_{\Omega_1} \neq 0$ . Then, by Theorem 1.2, the functional  $\|Tu_n - v\|_{\mathcal{Y}}^2 + \alpha_2 J_0^{\Omega_1}(u_n)$  is  $BV$ -coercive; i.e:

$$\|Tu_n - v\|_{\mathcal{Y}}^2 + \alpha_2 J_0^{\Omega_1}(u_n) \rightarrow \infty. \tag{23}$$

Now clearly

$$\begin{aligned}
\|Tu_n - v\|_{\mathcal{Y}}^2 + \alpha_2 J_0^{\Omega_1}(u_n) &\leq \|Tu_n - v\|_{\mathcal{Y}}^2 + \alpha_2 \sup_{\vec{v} \in \mathcal{V}_\theta} \int_{\Omega_0^c} -u_n \operatorname{div}(\theta \vec{v}) dx \\
&\leq F_\theta(u_n).
\end{aligned} \tag{24}$$

From (23) and (24) it follows that  $F_\theta(u_n) \rightarrow \infty$ . Hence  $F_\theta$  is  $L^2(\Omega)$ -coercive. The existence of a global minimizer then follows. Finally, the hypothesis that  $\mathcal{N}(T)$  does not contain functions vanishing on  $\Omega_1$  also implies that  $F_\theta(u)$  is strictly convex and therefore such a global minimizer is unique.  $\square$

### 3 Signal restoration with $L^2$ - $BV$ regularization

The purpose of this section is to show some applications of the regularization method developed in the previous section consisting in the simultaneous use of penalizers of  $L^2$  and of bounded-variation (BV) type to signal restoration problems.

A basic mathematical model for signal blurring is given by convolution, as a Fredholm integral equation of first kind:

$$v(t) = \int_0^1 k(t, s)u(s)ds, \quad (25)$$

where  $k(t, s)$  is the blurring kernel or point spread function,  $u$  is the exact (original) signal and  $v$  is the blurred signal. For the examples that follow we took a Gaussian blurring kernel, i.e.  $k(t, s) = \frac{1}{\sqrt{2\pi}\sigma_b} \exp\left(-\frac{(t-s)^2}{2\sigma_b^2}\right)$ , with  $\sigma_b > 0$ . Equation (25) was discretized in the usual way (using collocation and quadrature), resulting in a discrete model of the form

$$Af = g, \quad (26)$$

where  $A$  is a  $(n+1) \times (n+1)$  matrix,  $f, g \in \mathbb{R}^{n+1}$  ( $f_j = u(t_j)$ ,  $g_j = v(t_j)$ ,  $t_j = \frac{j}{n}$ ,  $0 \leq j \leq n$ ). We took  $n = 130$  and  $\sigma_b = 0.05$ . The data  $g$  was contaminated with a 1% zero-mean Gaussian additive noise (i.e. standard deviation equal to 1% of the range of  $g$ ).

**Example 3.1.** For this example, the original signal (unknown in real life problems) and the blurred noisy signal which constitutes the data of the inverse problem for this example are shown in Figure 1.

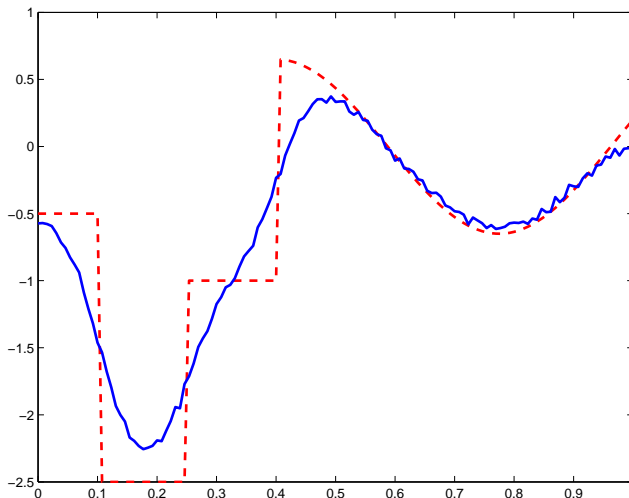


Figure 1: Original signal (---) and blurred noisy signal (—).

Figure 2 shows the regularized solutions obtained with the classical Tikhonov-Phillips method of order zero (left) and with penalizer associated to the bounded variation seminorm  $J_0$  (right). As expected, the regularized solution obtained with the  $J_0$  penalizer is significantly better than the one obtained with the classical Tikhonov-Phillips method near jumps and in regions where the exact solution is piecewise constant. The opposite happens where the exact solution is smooth.

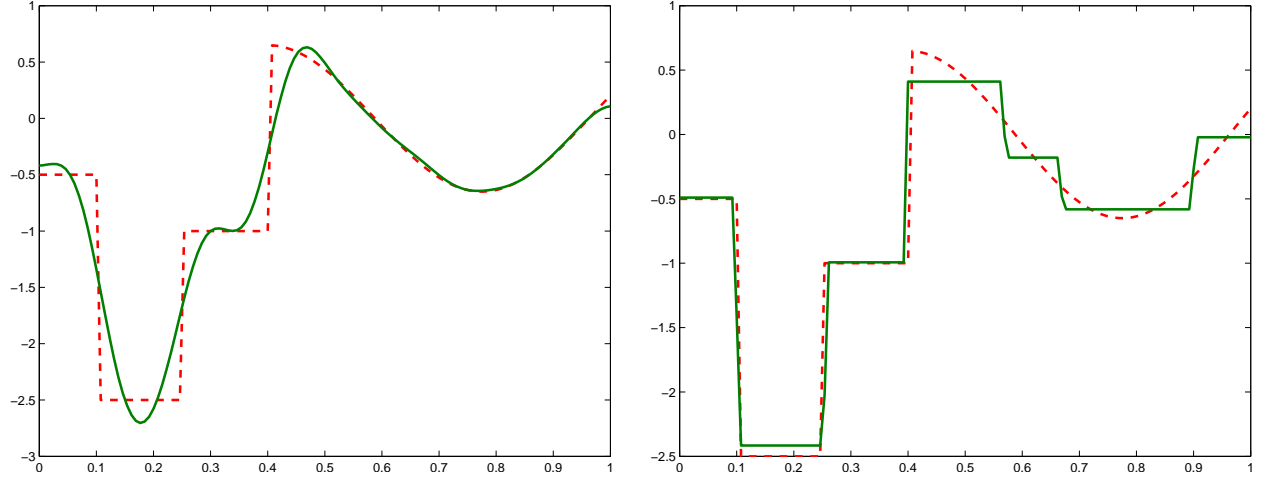


Figure 2: Original signal (---) and regularized solutions (—) obtained with Tikhonov-Phillips (left) and bounded variation seminorm (right).

Figure 3 shows the regularized solution obtained with the combined  $L^2$ –BV method (see (12)). In this case the weight function  $\theta(t)$  was chosen to be  $\theta(t) \doteq 1$  for  $t \in (0, 0.4]$  and  $\theta(t) \doteq 0$  for  $t \in (0.4, 1)$ . Although this choice of  $\theta(t)$  is clearly based upon “*a-priori*” information about the regularity of exact solution, other reasonable choices of  $\theta$  can be made by using only data-based information. Choosing a “good” weighting function  $\theta$  is a very important issue but we shall not discuss this matter in this article. For instance, one way of constructing a reasonable function  $\theta$  is by computing the normalized (in  $[0, 1]$ ) convolution of a Gaussian function of zero mean and standard deviation  $\sigma_b$  and the modulus of the gradient of the regularized solution obtained with a pure zero-order Tikhonov-Phillips method (see Figure 4). For this weight function  $\theta$ , the corresponding regularized solution obtained with the combined  $L^2$ –BV method is shown in Figure 5. In all cases reflexive boundary conditions were used ([9]) and the regularization parameters were calculated using Morozov’s discrepancy principle with  $\tau = 1.1$  ([7]).

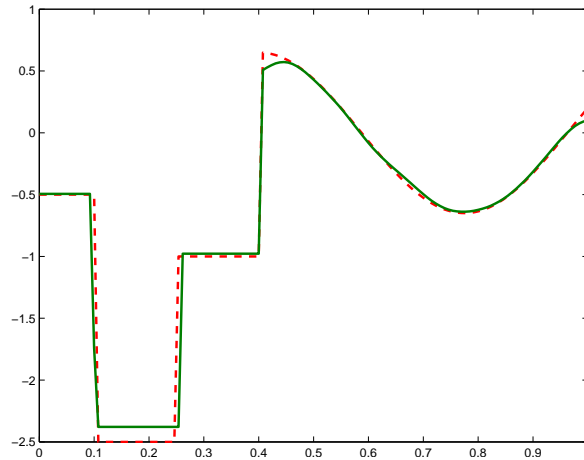


Figure 3: Original signal (---) and regularized solution (—) obtained with the combined  $L^2$ –BV method and binary weight function  $\theta$ .

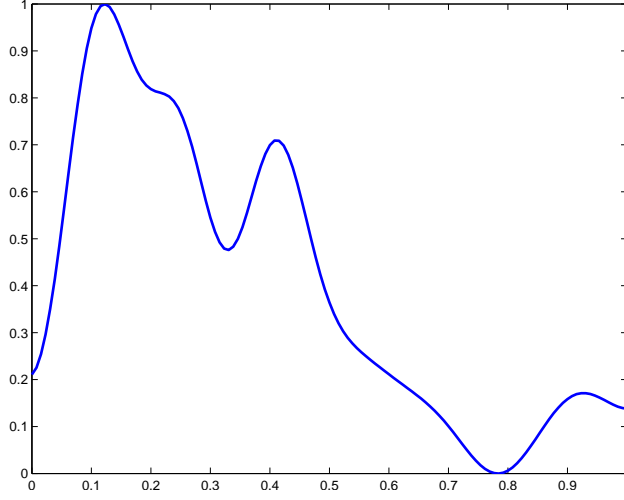


Figure 4: Weight function  $\theta$  computed by normalizing the convolution of a Gaussian kernel and the modulus of the gradient of the regularized solution with a pure Tikhonov-Phillips method.

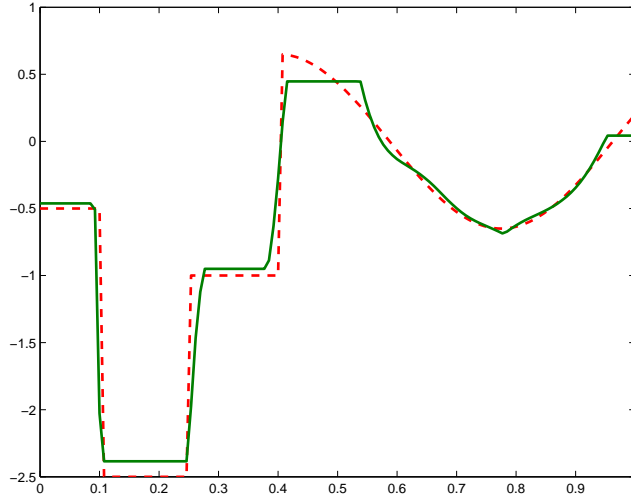


Figure 5: Original signal (---) and regularized solution (—) obtained with the combined  $L^2 - BV$  method and the data-based weight function  $\theta$  showed in Fig. 4.

As it can be seen, the improvement of the result obtained with the combined  $L^2 - BV$  method and “*ad-hoc*” binary function  $\theta$  with respect to the pure simple methods, zero-order Tikhonov-Phillips and pure  $BV$ , is notorious. As previously mentioned however, in this case the construction of the function  $\theta$  is based on “*a-priori*” information about the exact solution, which most likely will not be available in concrete real life problems. Nevertheless, the regularized solution obtained with the data-based weight function  $\theta$  shown in Figure 4 is also significantly better than those obtained with any of the single-based penalizers. This fact is clearly and objectively reflected by the Improved Signal-to-Noise Ratio (ISNR) defined as

$$ISNR = 10 \log_{10} \left( \frac{\|f - g\|^2}{\|f - f_\alpha\|^2} \right),$$

where  $f_\alpha$  is the restored signal obtained with regularization parameter  $\alpha$ . For all the previously shown restorations, the ISNR was computed in order to have a parameter for objectively measuring and comparing the quality of the regularized solutions (see Table 1).

Table 1: ISNR's for Example 3.1.

Regularization Method	ISNR
Tikhonov-Phillips of order zero	2.5197
Bounded variation seminorm	4.2063
Mixed $L^2$ -BV method with binary $\theta$	5.7086
Mixed $L^2$ -BV method with zero-order Tikhonov-based $\theta$	4.4029

**Example 3.2.** For this example we considered a signal which is smooth in two disjoint intervals and it is piecewise constant in their complement, having three jumps. The signal was blurred and noise was added just as in the previous example. The original and blurred-noisy signal are depicted in Figure 6.

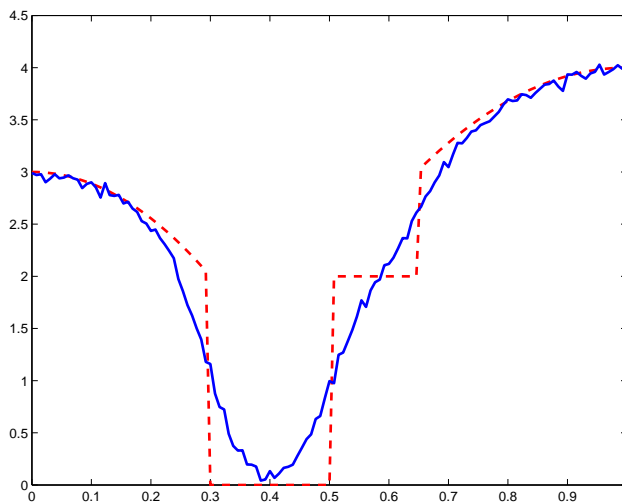


Figure 6: Original ( - - ) and blurred-noisy ( — ) signals for Example 3.2.

Figure 7 shows the restorations obtained with the classical zero-order Tikhonov-Phillips method (left) and  $BV$  with penalizer  $J_0$  (right).

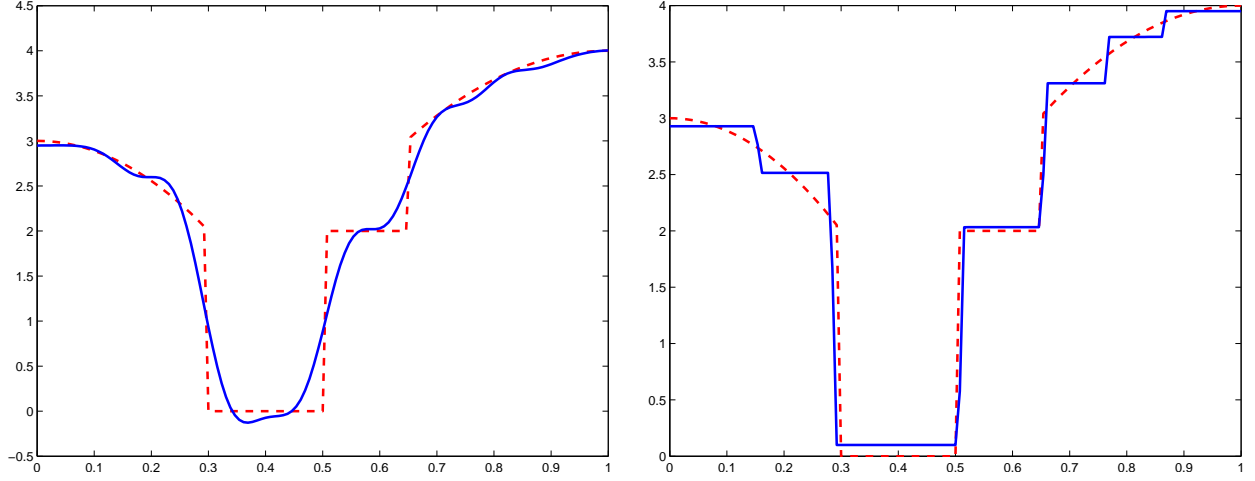


Figure 7: Original signal (---) and regularized solutions (—) obtained with Tikhonov-Phillips (top) and bounded variation seminorm (bottom).

An ad-hoc binary weight function  $\theta$  for this example was defined on the interval  $[0, 1]$  as  $\theta(t) = \chi_{[0.3, 0.65]}(t)$ . The regularized solution obtained with this weight function and the combined  $L^2 - BV$  method is shown in Figure 8. Once again, the improvement with respect to any of the classical pure methods is clearly notorious.

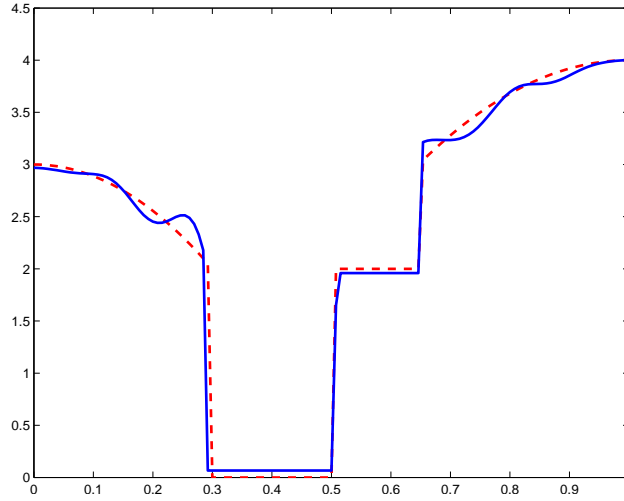


Figure 8: Original signal (---) and regularized solution (—) obtained with the combined  $L^2 - BV$  method and binary function  $\theta$ .

Here also we constructed a data based weight function  $\theta$  as in Example 3.1, by convolving a Gaussian kernel with the modulus of the gradient of a Tikhonov regularized solution and normalizing the result. This weight function  $\theta$  is now depicted in Figure 9, while the corresponding restored signal is shown in Figure 10.



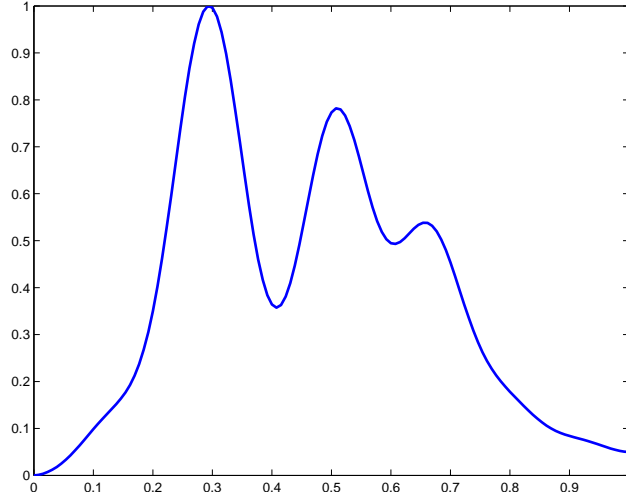


Figure 9: Tikhonov-based weight function  $\theta$  for Example 3.2.

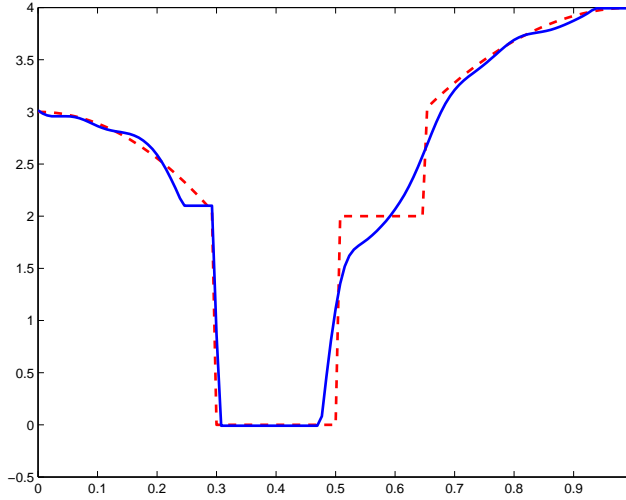


Figure 10: Original signal ( - - ) and regularized solution ( — ) obtained with the combined  $L^2 - BV$  method and function  $\theta$  showed in Fig. 9.

In table 2 the values of the ISNR for the four restorations are presented. These values show once again a significant improvement of the combined method with respect to any of the pure single methods.

Table 2: ISNR's for Example 3.2.

Regularization Method	ISNR
Tikhonov-Phillips of order zero	2.6008
Bounded variation seminorm	2.8448
Mixed $L^2 - BV$ method with binary $\theta$	4.8969
Mixed $L^2 - BV$ method with zero-order Tikhonov-based $\theta$	4.3315

## 4 Conclusions

In this article we introduced a new generalized Tikhonov-Phillips regularization method in which the penalizer is given by a spatially varying combination of the  $L^2$  norm and of the bounded variation seminorm. For particular cases, existence and uniqueness of global minimizers of the corresponding functionals were shown. Finally, applications of the new method to signal restoration problem were shown.

Although these preliminary results are clearly quite promising, further research is needed. In particular, the choice or construction of a weight function  $\theta(t)$  in a somewhat optimal way is a matter which undoubtedly deserves much further attention and study. Research in these directions is currently under way.

## Acknowledgments

This work was supported in part by Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET, through PIP 2010-2012 Nro. 0219, by Agencia Nacional de Promoción Científica y Tecnológica, ANPCyT, through project PICT 2008-1301, by Universidad Nacional del Litoral, through projects CAI+D 2009-PI-62-315, CAI+D PJov 2011 Nro. 50020110100055, CAI+D PI 2011 Nro. 50120110100294 and by the Air Force Office of Scientific Research, AFOSR, through Grant FA9550-10-1-0018.

## References

- [1] R. Acar and C. R. Vogel, *Analysis of bounded variation penalty methods for ill-posed problems*, Inverse Problems **10** (1994), 1217–1229.
- [2] R. A. Adams, *Sobolev spaces*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 65.
- [3] H. Attouch, G. Buttazzo, and G. Michaille, *Variational analysis in Sobolev and BV spaces*, MPS/SIAM Series on Optimization, vol. 6, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2006, Applications to PDEs and optimization.
- [4] P. Blomgren, T. F. Chan, P. Mulet, and C. Wong, *Total variation image restoration: Numerical methods and extensions*, Proceedings of the IEEE International Conference on Image Processing **III** (1997), 384–387.
- [5] A. Chambolle and J. L. Lions, *Image recovery via total variation minimization and related problems*, Numer. Math. **76** (1997), 167–188.
- [6] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM Journal Applied Mathematical **66** (2006), no. 4, 1383–1406.
- [7] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of inverse problems*, Mathematics and its Applications, vol. 375, Kluwer Academic Publishers Group, Dordrecht, 1996.
- [8] J. Hadamard, *Sur les problèmes aux dérivées partielles et leur signification physique*, Princeton University Bulletin **13** (1902), 49–52.

- [9] P. C. Hansen, *Discrete inverse problems: Insight and algorithms*, Fundamentals of Algorithms, vol. FA07, Society for Industrial and Applied Mathematics, Philadelphia, 2010.
- [10] F Li, Z. Li, and L. Pi, *Variable exponent functionals in image restoration*, Applied Mathematics and Computation **216** (2010), 870–882.
- [11] G. L. Mazziari, R. D. Spies, and K. G. Temperini, *Existence, uniqueness and stability of minimizers of generalized tikhonov-phillips functionals*, Journal of Mathematical Analysis and Applications **396** (2012), 396–411.
- [12] L. I. Rudin, S. Osher, and Fatemi E., *Nonlinear total variation based noise removal algorithms (proceedings of the 11th annual international conference of the center for nonlinear studies)*, Physica D **60** (1992), 259–268.